Linear Convergence of Natural Policy Gradient Methods with Log-Linear Policies

Rui Yuan[1,4], Simon S. Du[2], Robert M. Gower[3], Alessandro Lazaric[1], Lin Xiao[1]

[1]Meta AI, [2]University of Washington, [3]Flatiron Institute, [4]Télécom Paris

International Conference on Learning Representations (ICLR), 2023

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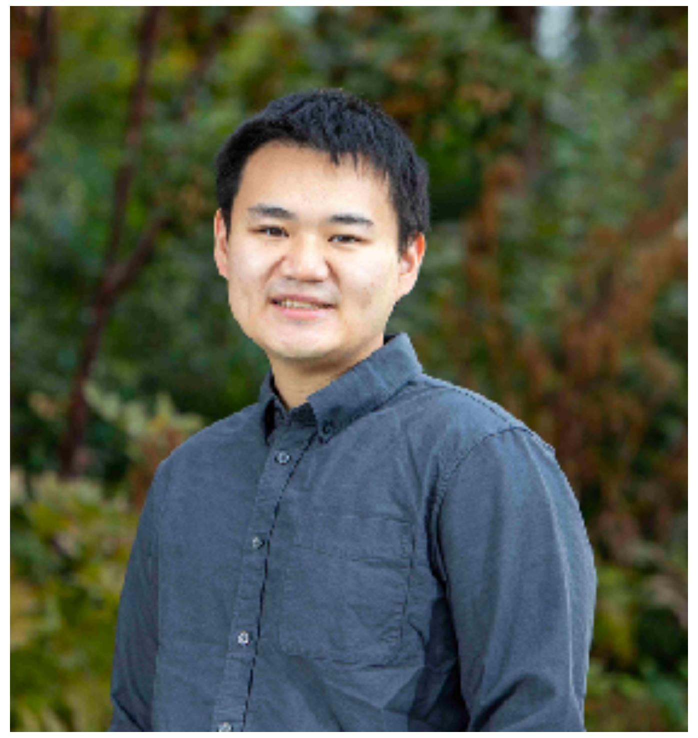


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Impressive Reinforcement Learning Results

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Board Game

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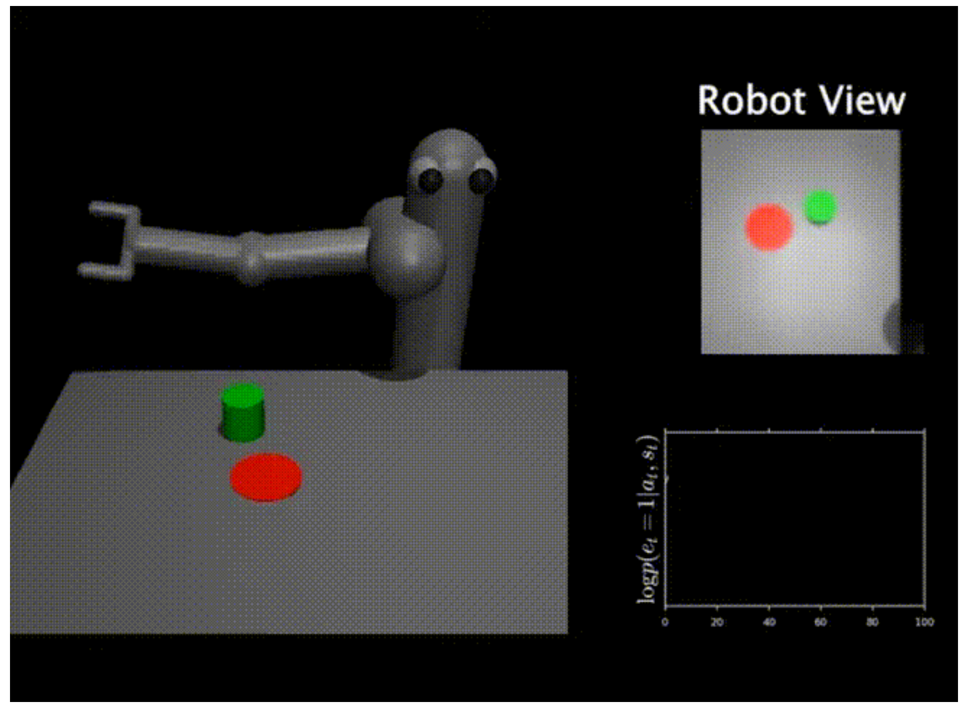
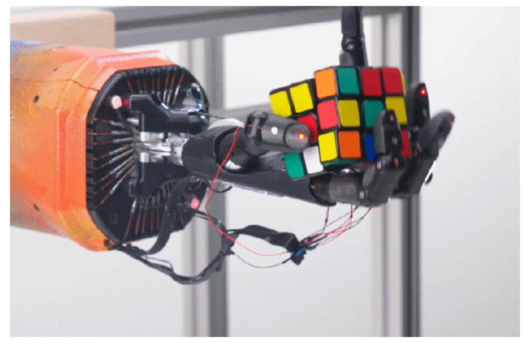
Robotic Manipulation





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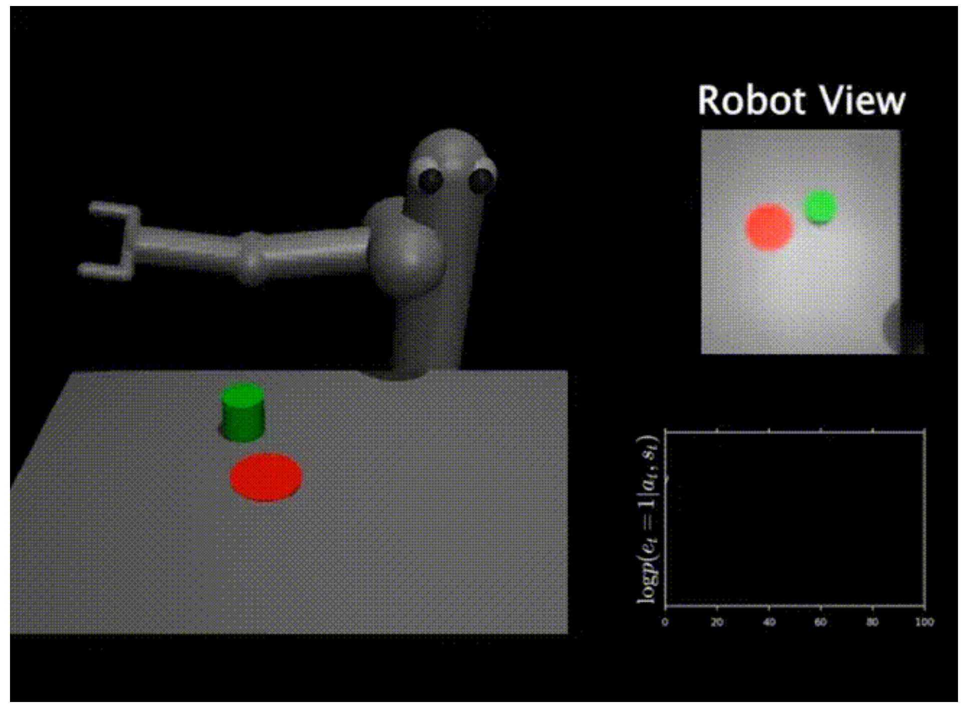
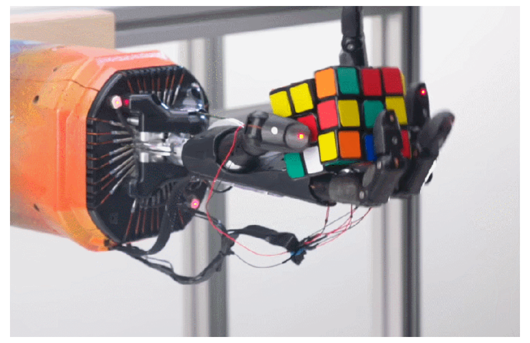
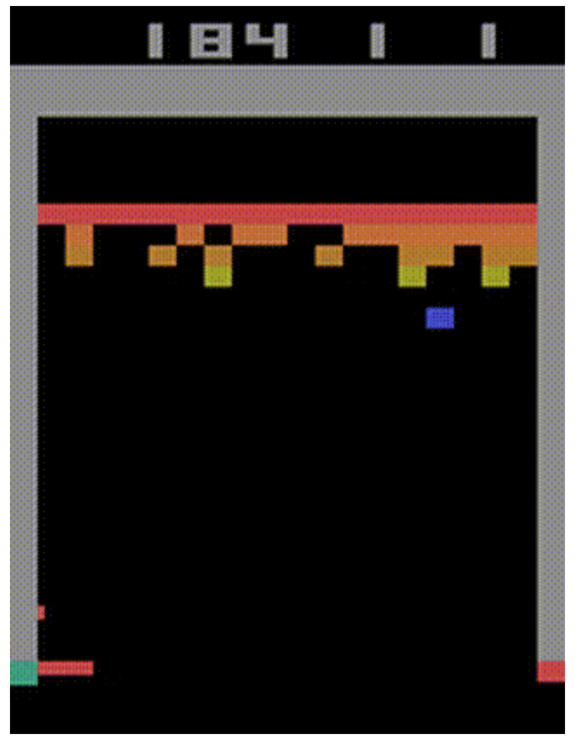
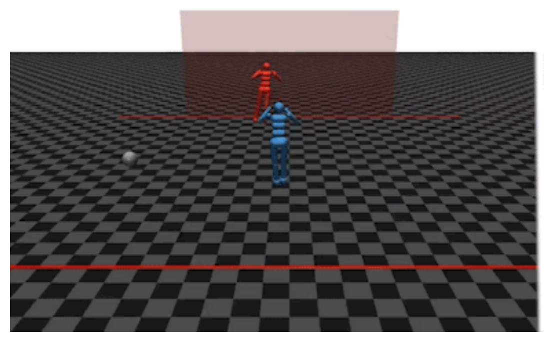
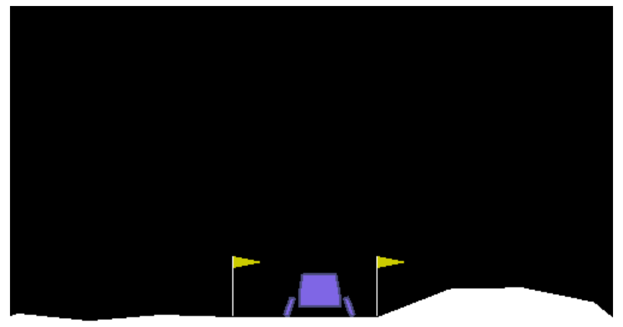




Game Playing







Reinforcement Learning (RL)

Sequential decision making problems

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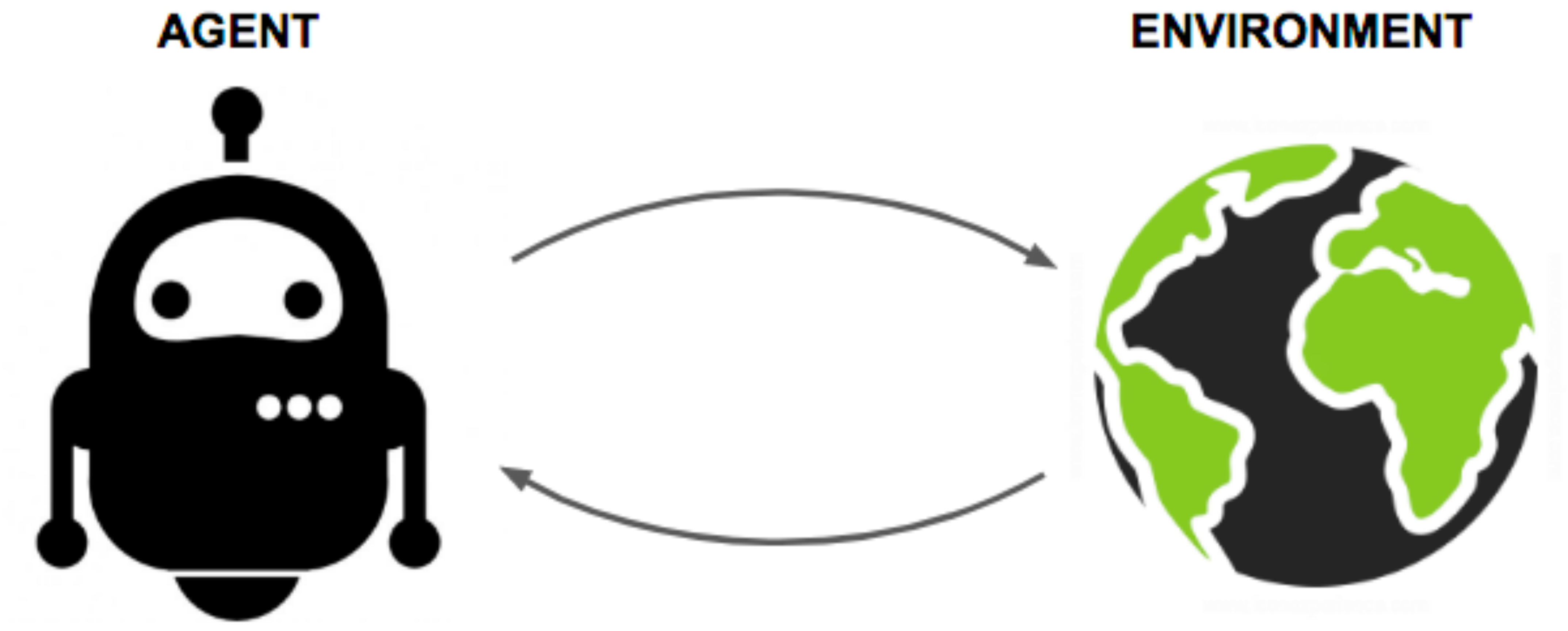

Markov decision Process (MDP)

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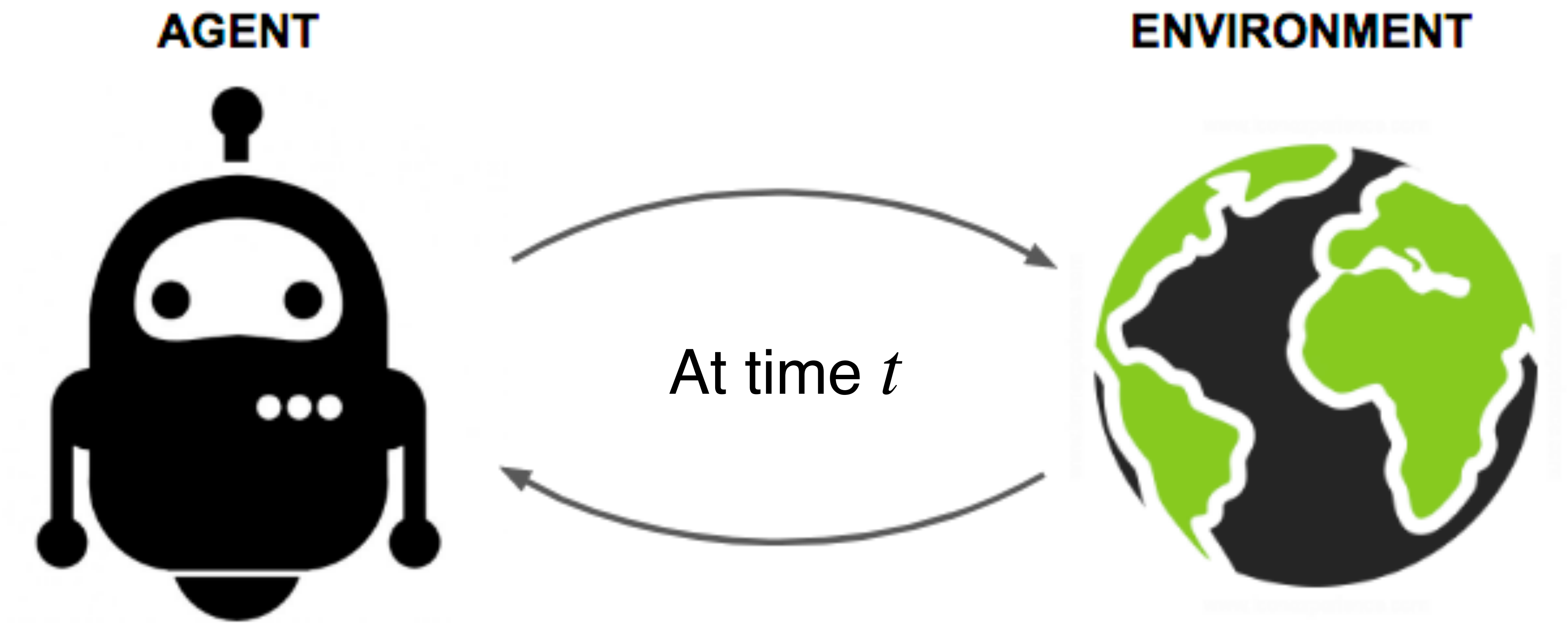

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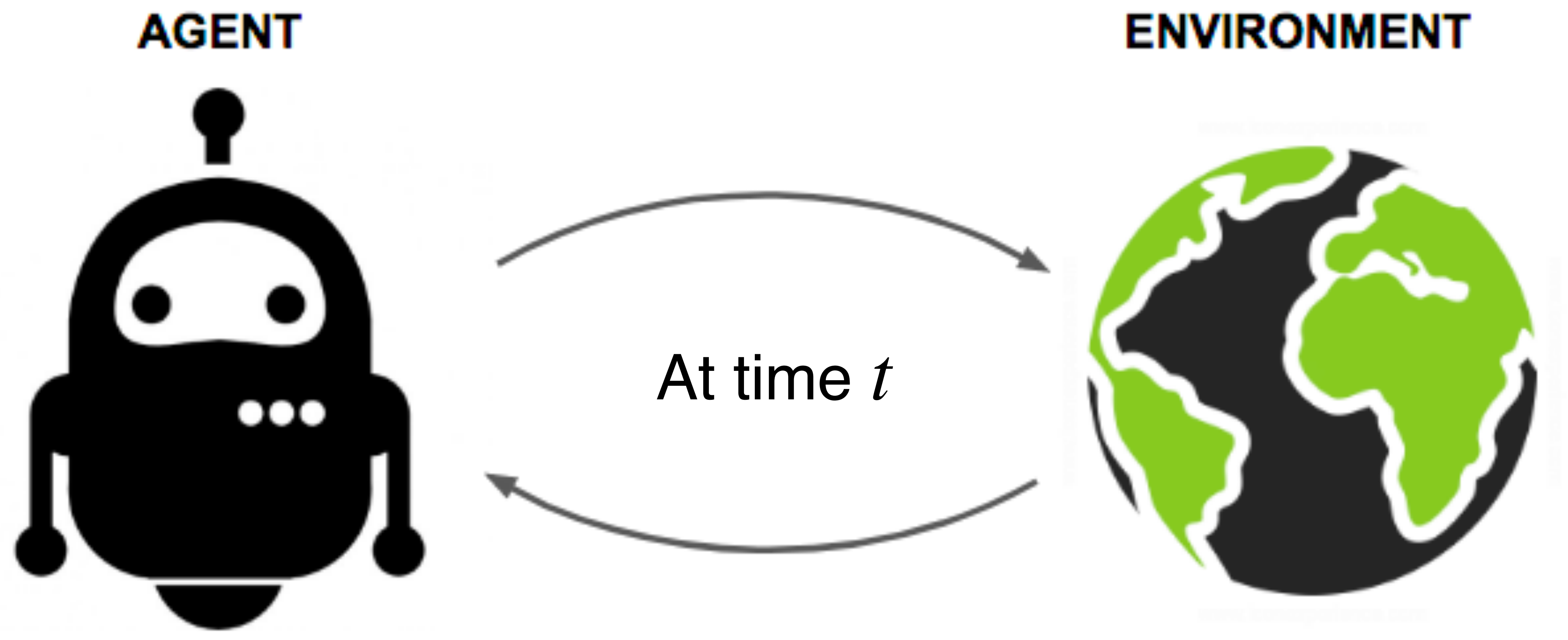

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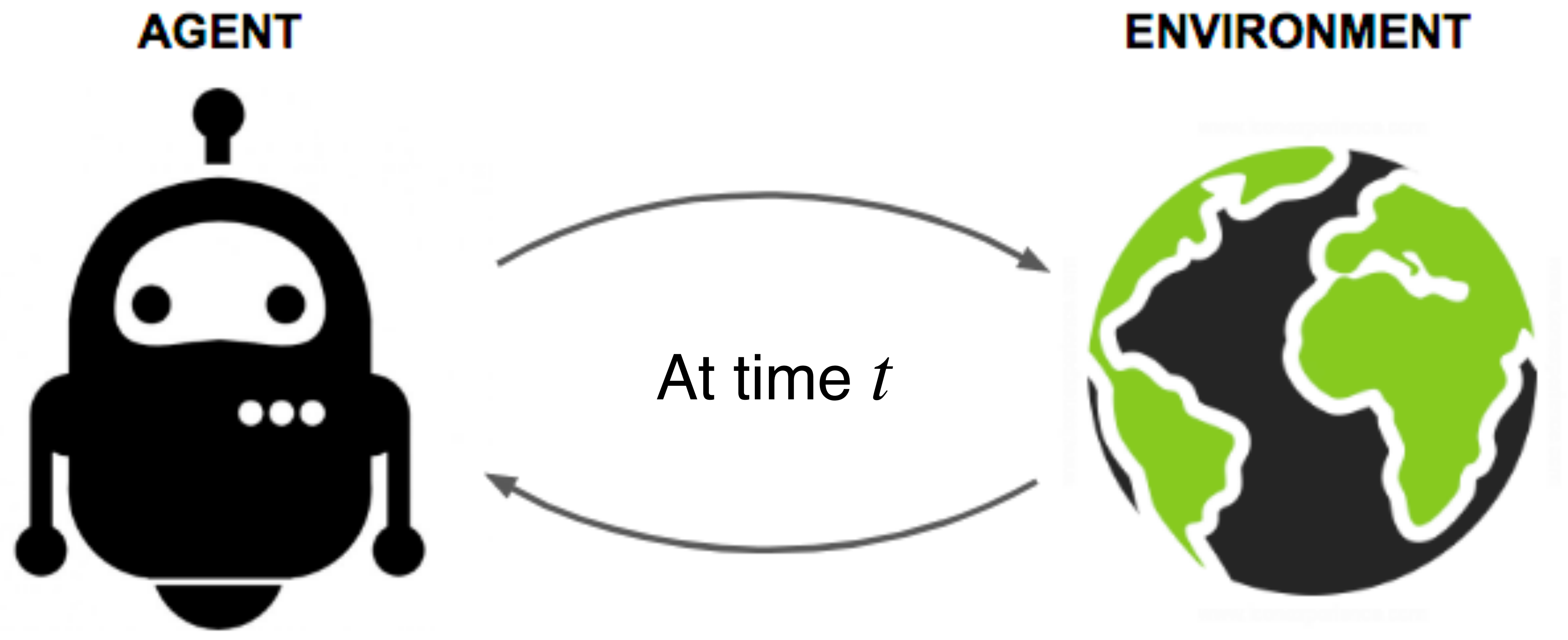

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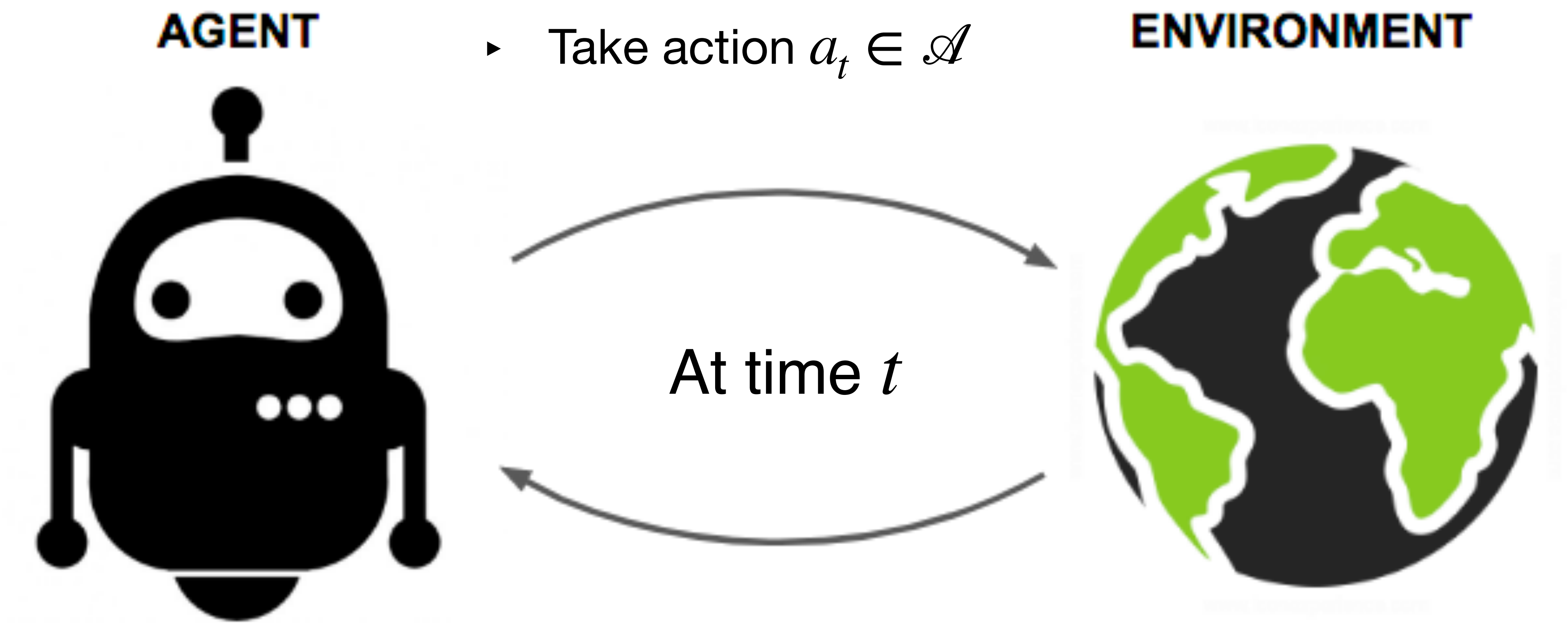

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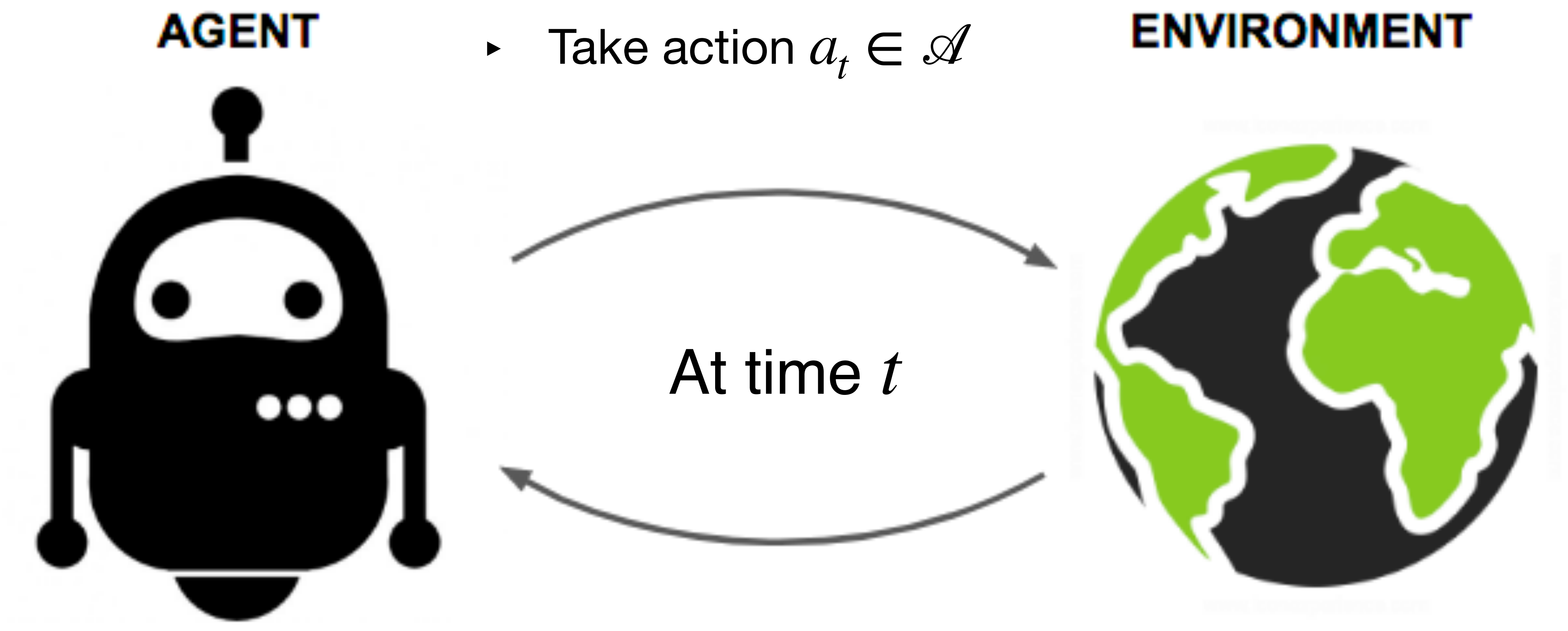

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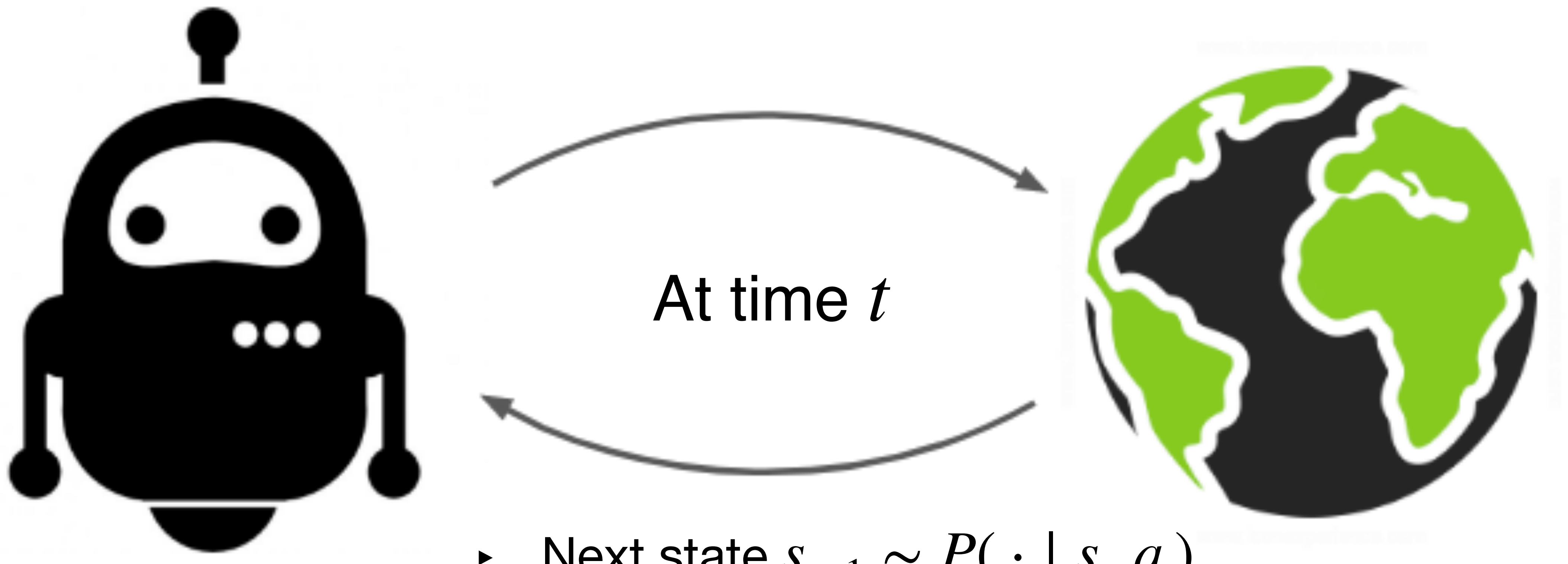

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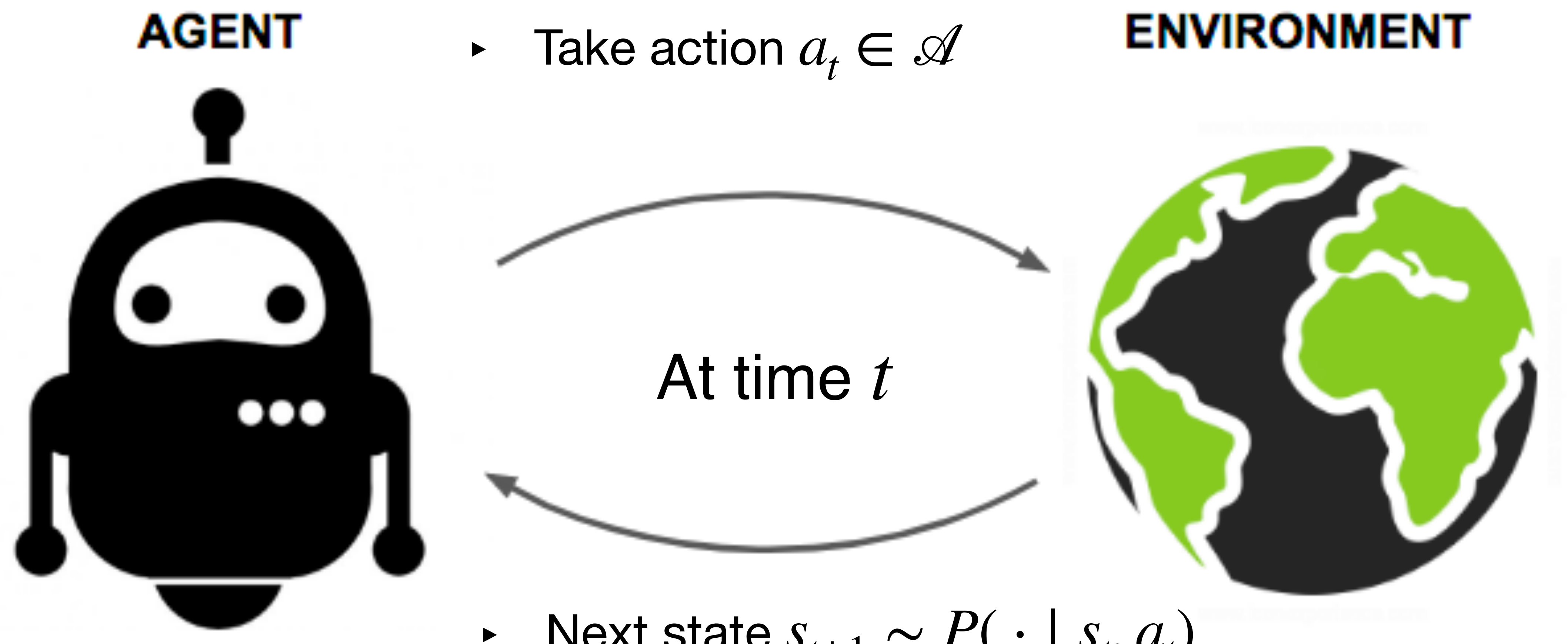

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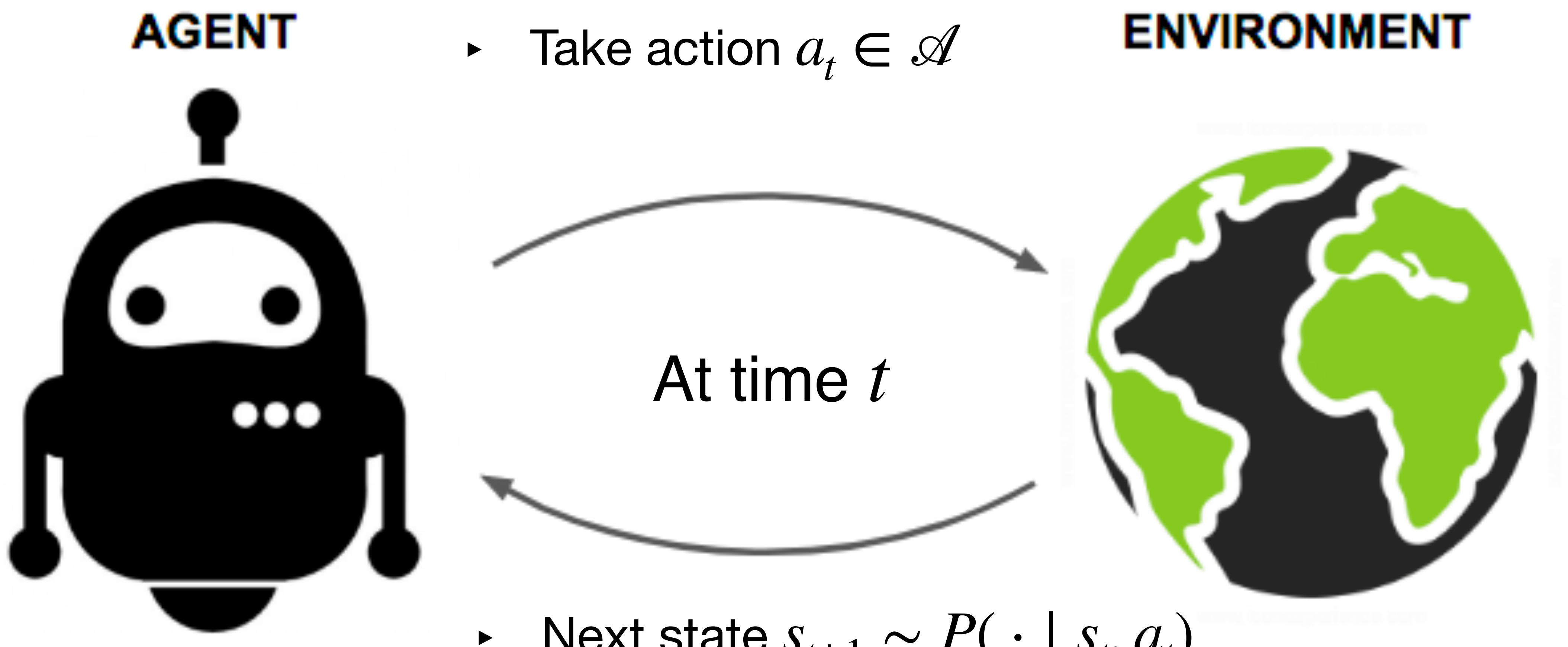

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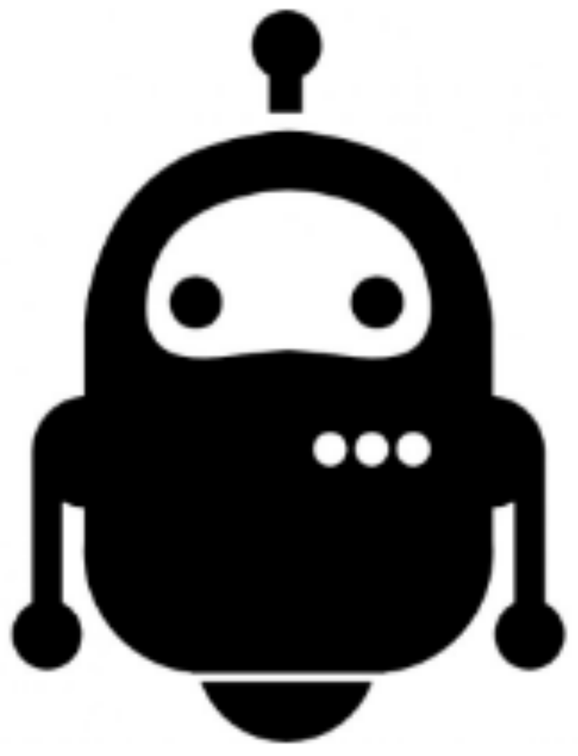

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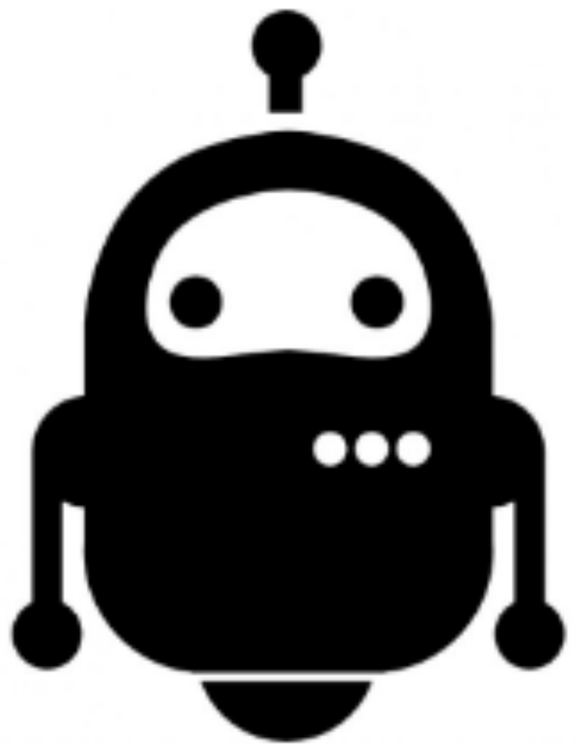

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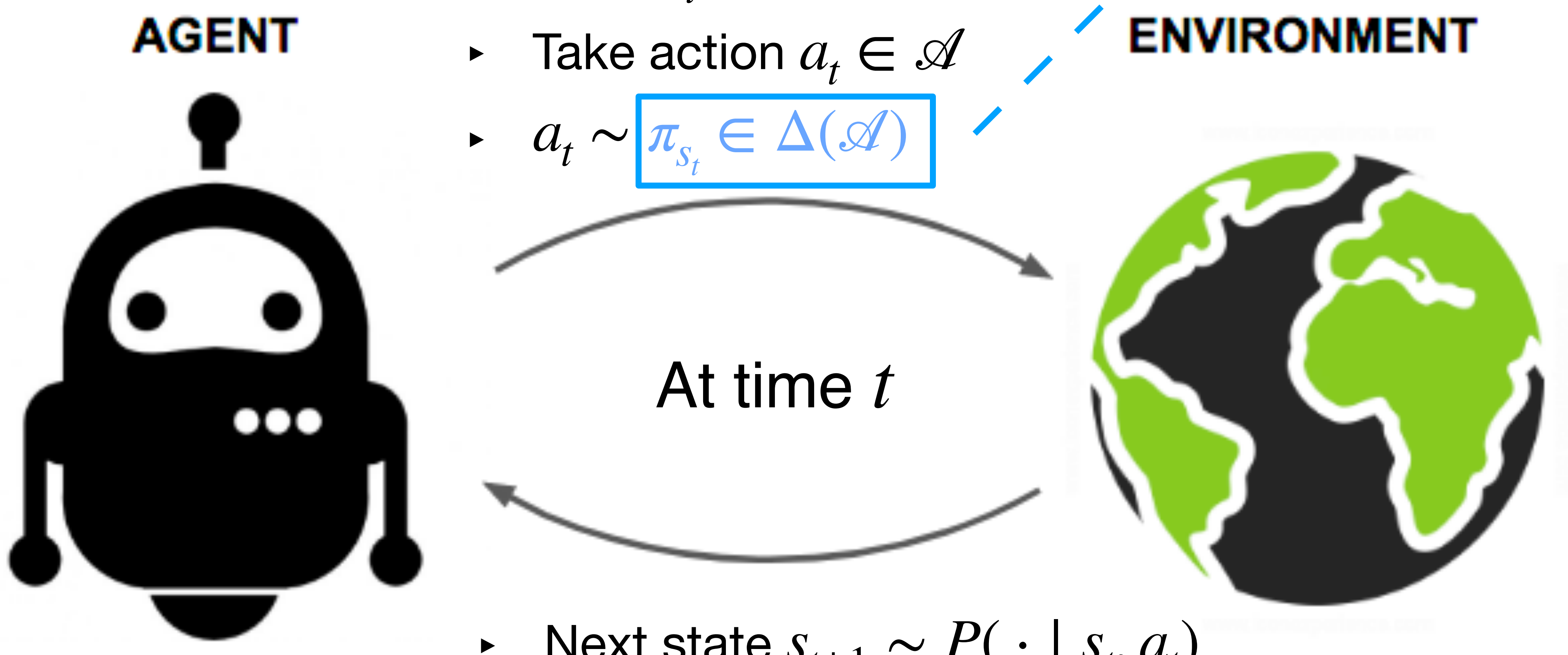

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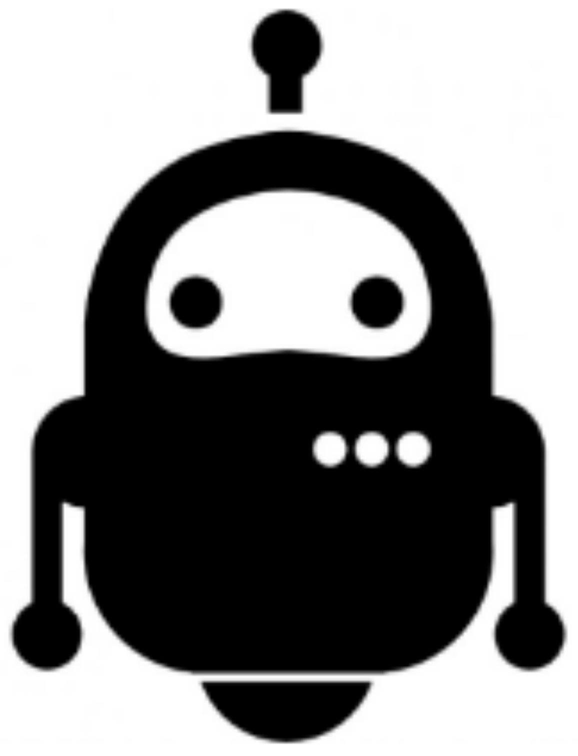

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$$\arg\min_{\pi} V_\rho(\pi) := \mathbb{E}_{s_0\sim\rho,\ a_t\sim\pi_{s_t},\ s_{t+1}\sim P(\cdot|s_t,a_t)}\left[\sum_{t=0}^{\infty} \gamma^t c(s_t, a_t)\right]$$

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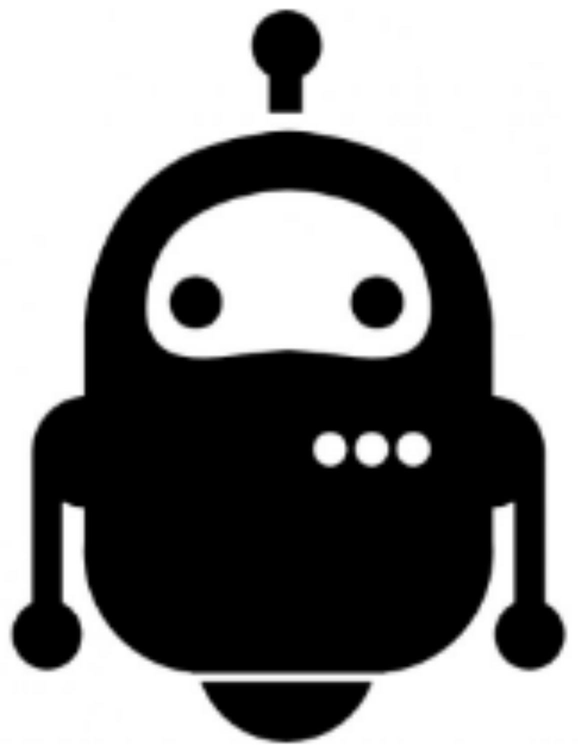

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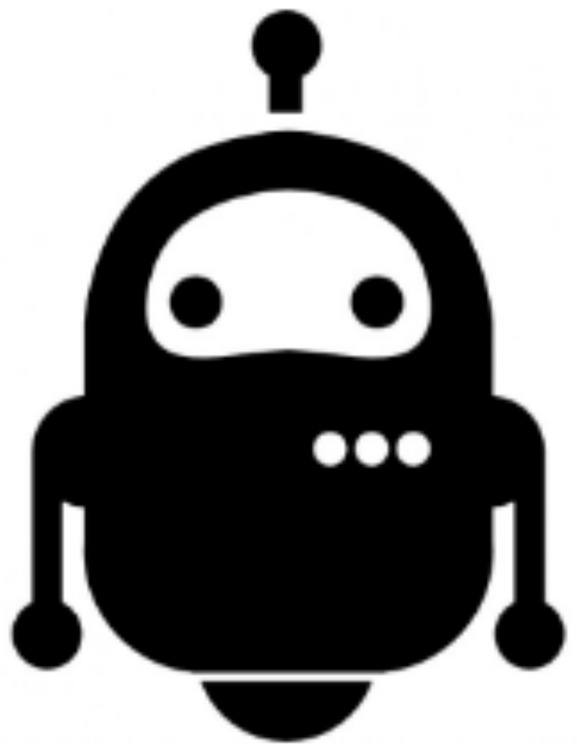

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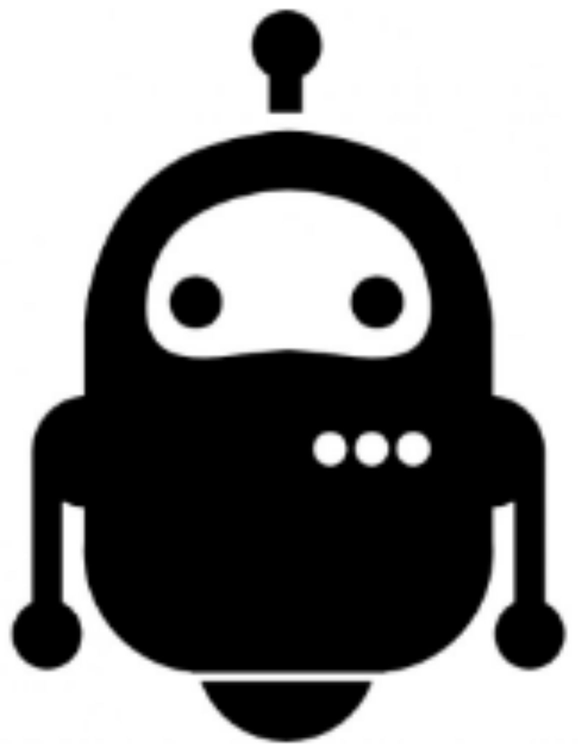

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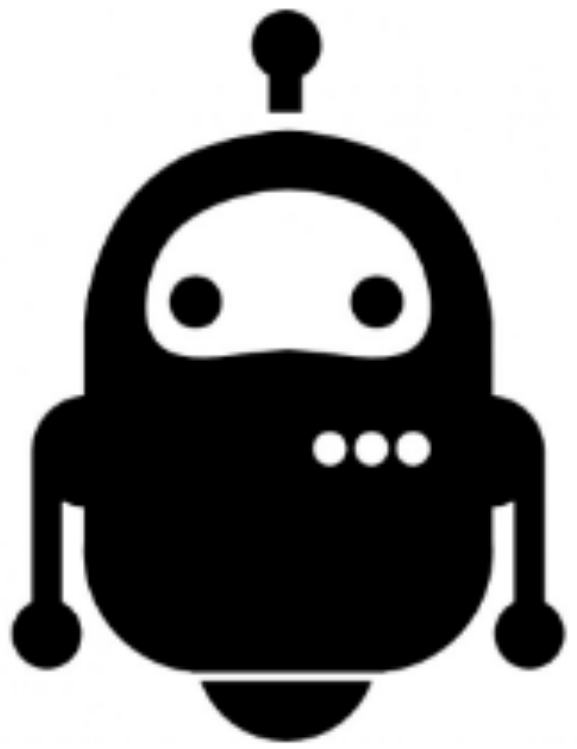

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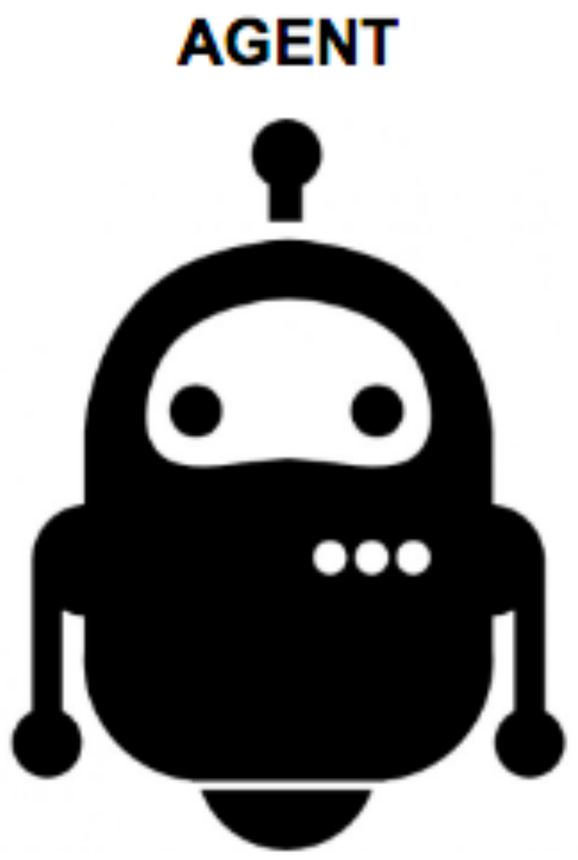

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Gradient of $V_\rho(\theta)$

Step size

- Simplicity

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  - Trust-region (e.g. TRPO [Schulman et al., 2015;]), proximal (e.g. PPO [Schulman et al., 2017])

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Objective: $\arg\min_{\theta \in \mathbb{R}^d} V_\rho(\theta)$

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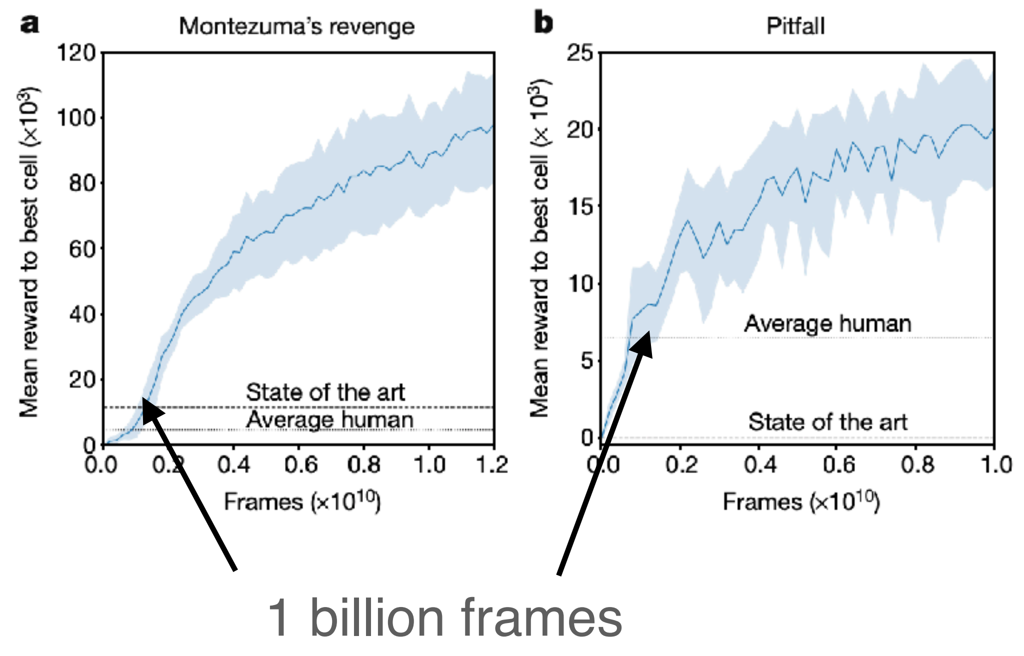

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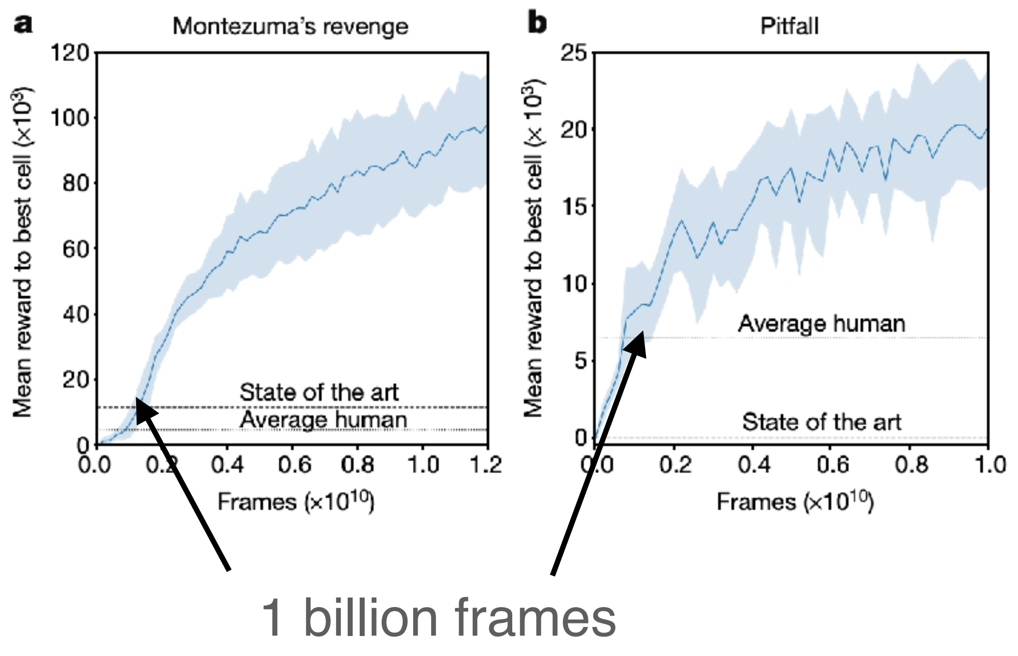

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- Natural PG (NPG)[Kakade, 2001] uses a preconditioner to improve PG direction

Context

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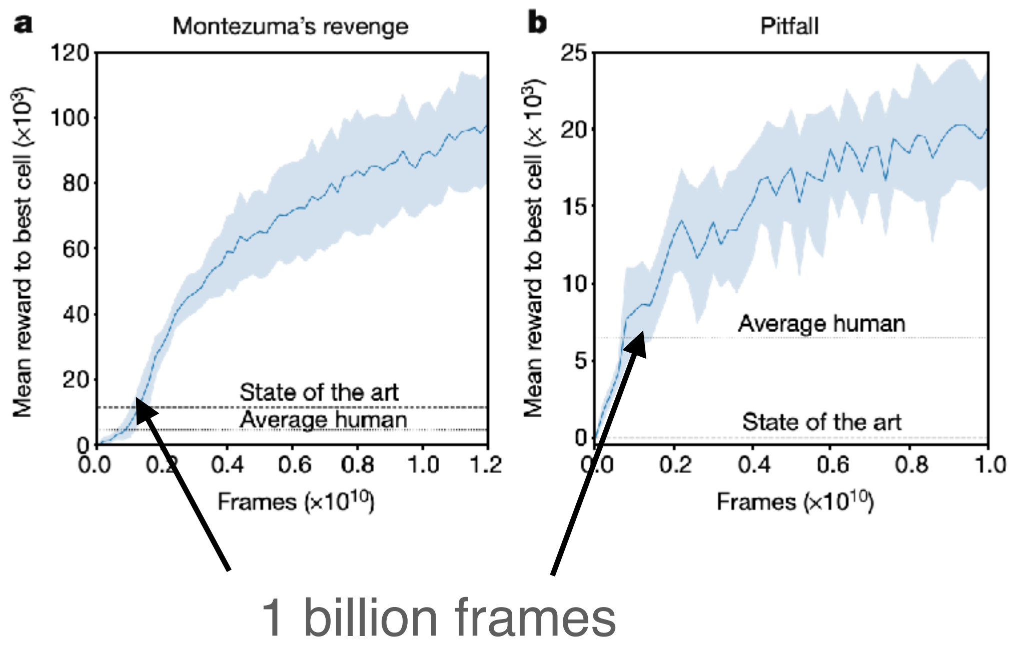

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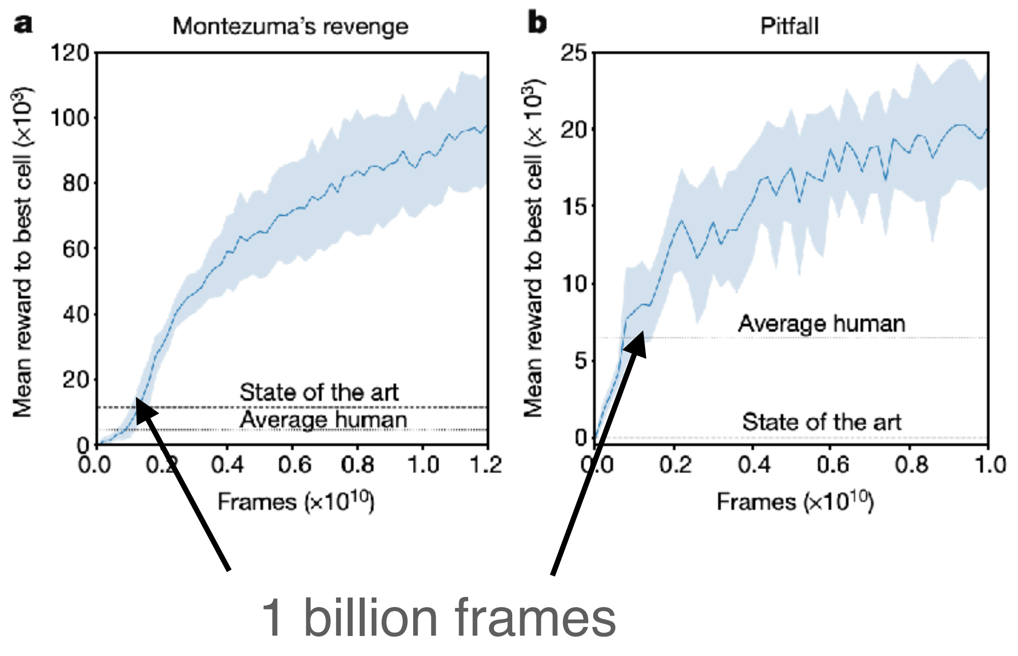

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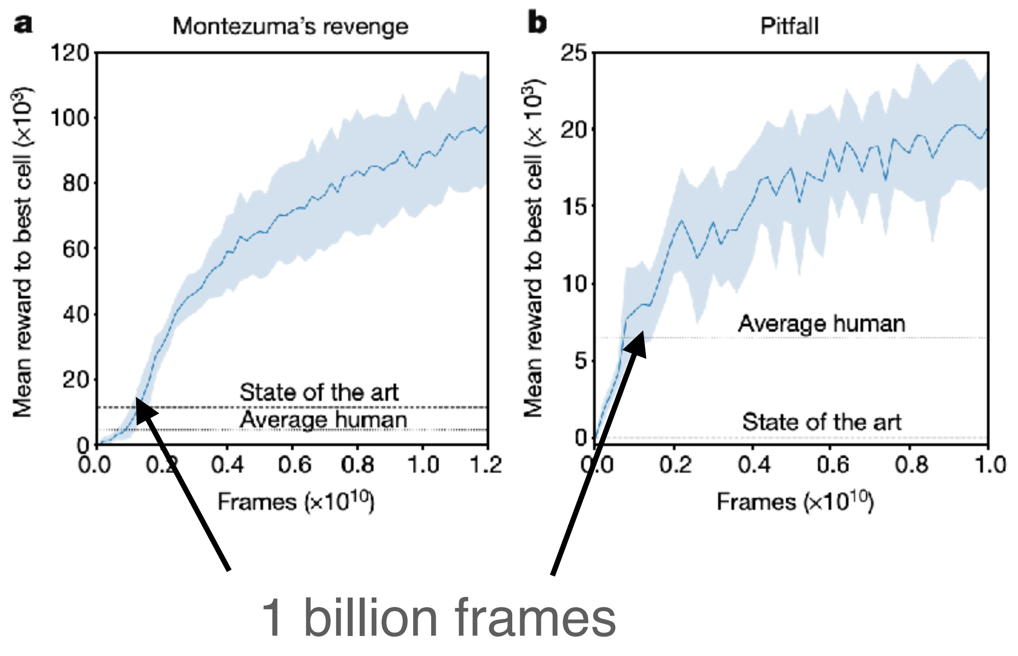

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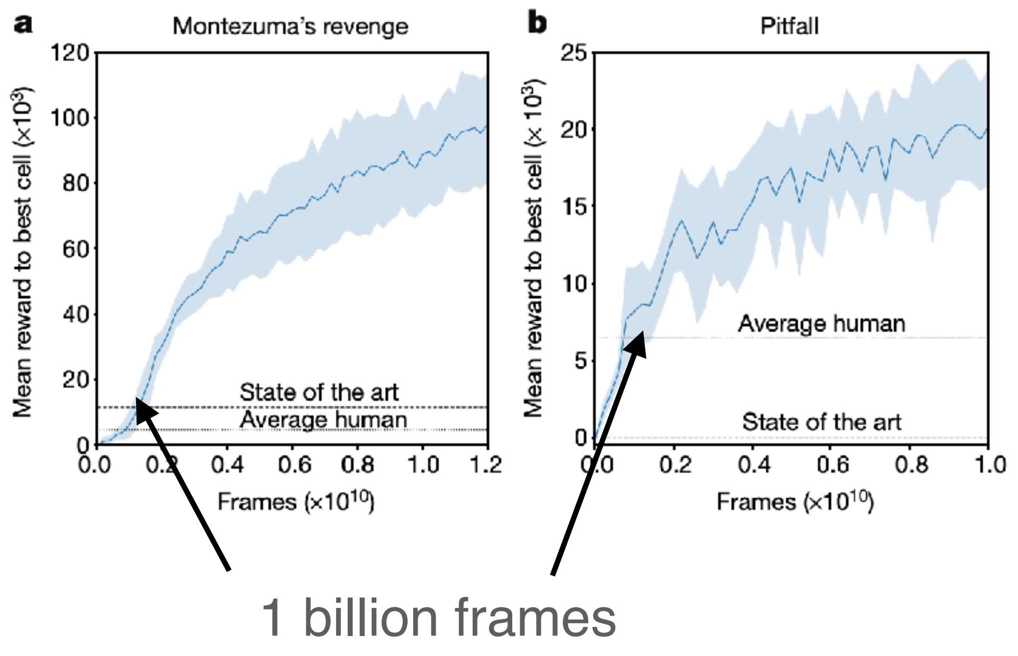

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Motivations

- Extend linear convergence of NPG from tabular to function approximation regime.

Natural policy gradient

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- The linear convergence analysis of NPG with log-linear policy can be extended to general parametrization [A Novel Framework for Policy Mirror Descent with General Parametrization and Linear Convergence, Carlo Alfano, Rui Yuan, and Patrick Rebeschini, 2023].

Thank you !

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- *Proof:*

$$\begin{aligned}
\nabla_\theta V_\rho(\theta) &= \nabla_\theta \sum\nolimits_{s_0\in\mathcal{S},\ a_0\in\mathcal{A}} \rho(s_0)\pi_{s_0,a_0}(\theta)Q_{s_0,a_0}(\theta) \\
&= \sum\nolimits_{s_0,a_0} \rho(s_0)\left(\nabla_\theta \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) + \sum\nolimits_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta Q_{s_0,a_0}(\theta) \\
&= \sum\nolimits_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \sum\nolimits_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta\left(c(s_0,a_0) + \gamma\sum\nolimits_{s_1} P(s_1 \mid s_0,a_0)V_{s_1}(\theta)\right)
\end{aligned}$$

Derivation of Policy Gradient Theorem

$$\nabla_\theta V_\rho(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi(\theta)}(\rho), a\sim\pi_s(\theta)}\left[Q_{s,a}(\theta)\,\nabla_\theta \log \pi_{s,a}(\theta)\right]$$

- *Proof:*

$$\begin{aligned}
\nabla_\theta V_\rho(\theta) &= \nabla_\theta \sum_{s_0\in\mathcal{S},\ a_0\in\mathcal{A}} \rho(s_0)\pi_{s_0,a_0}(\theta)Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\left(\nabla_\theta \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta\left(\boxed{c(s_0,a_0) + \gamma\sum_{s_1} P(s_1 \mid s_0,a_0)V_{s_1}(\theta)}\right)
\end{aligned}$$

Bellman Equation

Derivation of Policy Gradient Theorem

$$\nabla_\theta V_\rho(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi(\theta)}(\rho), a\sim\pi_s(\theta)}\left[Q_{s,a}(\theta)\,\nabla_\theta \log \pi_{s,a}(\theta)\right]$$

- *Proof:*

$$\begin{aligned}
\nabla_\theta V_\rho(\theta) &= \nabla_\theta \sum_{s_0\in\mathcal{S},\ a_0\in\mathcal{A}} \rho(s_0)\pi_{s_0,a_0}(\theta)Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\left(\nabla_\theta \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta\left(\boxed{c(s_0,a_0) + \gamma\sum_{s_1} P(s_1 \mid s_0,a_0)V_{s_1}(\theta)}\right) \quad \text{Bellman Equation} \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \gamma\sum_{s_0,a_0,s_1} \rho(s_0)\pi_{s_0,a_0}(\theta)P(s_1 \mid s_0,a_0)\,\nabla_\theta V_{s_1}(\theta)
\end{aligned}$$

Derivation of Policy Gradient Theorem

$$\nabla_\theta V_\rho(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi(\theta)}(\rho), a\sim\pi_s(\theta)}\left[Q_{s,a}(\theta)\,\nabla_\theta \log \pi_{s,a}(\theta)\right]$$

- *Proof:*

$$\begin{aligned}
\nabla_\theta V_\rho(\theta) &= \nabla_\theta \sum_{s_0\in\mathcal{S},\ a_0\in\mathcal{A}} \rho(s_0)\pi_{s_0,a_0}(\theta)Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\left(\nabla_\theta \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta\left(\boxed{c(s_0,a_0) + \gamma\sum_{s_1} P(s_1 \mid s_0,a_0)V_{s_1}(\theta)}\right) \quad \text{Bellman Equation} \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \gamma\sum_{s_0,a_0,s_1} \rho(s_0)\pi_{s_0,a_0}(\theta)P(s_1 \mid s_0,a_0)\,\nabla_\theta V_{s_1}(\theta) \\
&= \mathbb{E}\left[Q_{s_0,a_0}(\theta)\,\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right] + \gamma\mathbb{E}\left[\nabla_\theta V_{s_1}(\theta)\right]
\end{aligned}$$

Derivation of Policy Gradient Theorem

$$\nabla_\theta V_\rho(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi(\theta)}(\rho),a\sim\pi_s(\theta)}\left[Q_{s,a}(\theta)\,\nabla_\theta \log \pi_{s,a}(\theta)\right]$$

- *Proof:*

$$\begin{aligned}
\nabla_\theta V_\rho(\theta) &= \nabla_\theta \sum_{s_0\in\mathcal{S},\ a_0\in\mathcal{A}} \rho(s_0)\pi_{s_0,a_0}(\theta)Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\left(\nabla_\theta \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta\left(\boxed{c(s_0,a_0) + \gamma\sum_{s_1} P(s_1 \mid s_0,a_0)V_{s_1}(\theta)}\right) \quad \text{Bellman Equation} \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right) Q_{s_0,a_0}(\theta) \\
&\quad + \gamma \sum_{s_0,a_0,s_1} \rho(s_0)\pi_{s_0,a_0}(\theta)P(s_1 \mid s_0,a_0)\,\nabla_\theta V_{s_1}(\theta) \\
&= \mathbb{E}\left[Q_{s_0,a_0}(\theta)\,\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right] + \gamma\mathbb{E}\left[\nabla_\theta V_{s_1}(\theta)\right] \\
&= \mathbb{E}\left[Q_{s_0,a_0}(\theta)\,\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right] + \gamma\mathbb{E}\left[Q_{s_1,a_1}(\theta)\,\nabla_\theta \log \pi_{s_1,a_1}(\theta)\right] + \cdots
\end{aligned}$$

Derivation of Policy Gradient Theorem

$$\nabla_\theta V_\rho(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi(\theta)}(\rho),a\sim\pi_s(\theta)}\left[Q_{s,a}(\theta)\,\nabla_\theta \log \pi_{s,a}(\theta)\right]$$

- *Proof:*

$$\begin{aligned}
\nabla_\theta V_\rho(\theta) &= \nabla_\theta \sum_{s_0\in\mathcal{S},\ a_0\in\mathcal{A}} \rho(s_0)\pi_{s_0,a_0}(\theta)Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\left(\nabla_\theta \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta Q_{s_0,a_0}(\theta) \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\,\nabla_\theta\left(\boxed{c(s_0,a_0) + \gamma\sum_{s_1} P(s_1 \mid s_0,a_0)V_{s_1}(\theta)}\right) \quad \text{Bellman Equation} \\
&= \sum_{s_0,a_0} \rho(s_0)\pi_{s_0,a_0}(\theta)\left(\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right)Q_{s_0,a_0}(\theta) \\
&\quad + \gamma\sum_{s_0,a_0,s_1} \rho(s_0)\pi_{s_0,a_0}(\theta)P(s_1 \mid s_0,a_0)\,\nabla_\theta V_{s_1}(\theta) \\
&= \mathbb{E}\left[Q_{s_0,a_0}(\theta)\,\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right] + \gamma\mathbb{E}\left[\nabla_\theta V_{s_1}(\theta)\right] \\
&= \mathbb{E}\left[Q_{s_0,a_0}(\theta)\,\nabla_\theta \log \pi_{s_0,a_0}(\theta)\right] + \gamma\mathbb{E}\left[Q_{s_1,a_1}(\theta)\,\nabla_\theta \log \pi_{s_1,a_1}(\theta)\right] + \cdots \\
&= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi(\theta)}(\rho),a\sim\pi_s(\theta)}\left[Q_{s,a}(\theta)\,\nabla_\theta \log \pi_{s,a}(\theta)\right]
\end{aligned}$$